

CLIQUEs IN THE UNION OF C_4 -FREE GRAPHS

ABEER OTHMAN AND ELI BERGER

ABSTRACT. Let B and R be two simple graphs with vertex set V , and let $G(B, R)$ be the simple graph with vertex set V , in which two vertices are adjacent if they are adjacent in at least one of B and R . We prove that if B and R are two C_4 -free graphs on the same vertex set V and $G(B, R)$ is the complete graph, then there exists an B -clique X , an R -clique Y and a clique Z in B and R , such that $V = X \cup Y \cup Z$. Further, if $x \in Z$ then x is one of the vertices of some double C_5 in $G(B, R)$. In particular, if also $G(B, R)$ does not contains a double C_5 , then V is obedient. We obtain that if B and R are C_4 -free graphs then $\omega(G(B, R)) \leq \omega(B) + \omega(R) + \frac{1}{2} \min(\omega(B), \omega(R))$ and $\omega(G(B, R)) \leq \omega(B) + \omega(R) + \omega(H(B, R))$ where $H(B, R)$ is the simple graph with vertex set V , in which two vertices are adjacent if they are adjacent in B and R .

1. INTRODUCTION

All graphs in this article are simple in the sense that they do not have double edges or loops. Let B (for “Blue”) and R (for “Red”) be two graphs on the same vertex set V . Denote by $G(B, R)$ (respectively, $H(B, R)$) the graph with vertex set V , in which two vertices are adjacent if they are adjacent in B or (respectively, and) in R . Recall that a **clique** in a graph is a set of pairwise adjacent vertices. If $G(B, R)$ is the complete graph, then it is naturally to ask: Does V is the union of a clique in B and a clique in R ?. Easier examples show that the answer is “No”. For example, if $B = C_5$ and R is its complement, then $\omega(B) + \omega(R) = 4 < |V| = 5$, where $\omega(G)$ is the maximal size of a clique in G . In particular, V is not the union of a clique in B and a clique in R . Some results showed that under some additional assumptions on the graphs B and R , we obtain that V is the union of a clique in B and a clique in R . We begin to summarize them. First, we need the following definitions.

Definition 1.1. We say that a subset $U \subseteq V$ is **obedient** if there exist an R -clique X and an B -clique Y such that $U = X \cup Y$.

Definition 1.2. A graph G is called **chordal** if each of its cycles of four or more nodes has a **chord**, which is an edge joining two nodes that are not adjacent in the cycle. In other words, G is C_k -**free** for $k \geq 4$, i.e., G has no induced cycles of length at least four.

In [3], Gyárfás and Lehel proved the following theorem.

Theorem 1.3 (Theorem 3 of [3]). *Let B and R be two graphs on the same vertex set V and assume that $G(B, R)$ is the complete graph. If B and R are C_k -free for $k = 4$ and $k = 5$, then V is obedient.*

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The methods used in [3] are combinatorial. Using topological methods, Berger [2] proved the following theorem.

Theorem 1.4 ([2]). *Let B and R be two graphs on the same vertex set and assume that $G(B, R)$ is the complete graph. If B is chordal and R is C_4 -free, then V is obedient.*

Recently, Aharoni, Berger, Chudnovsky and Ziani generalized Theorems 1.3 and 1.4 as follow.

Theorem 1.5 (Theorem 1.7 of [1]). *Let B and R be C_4 -free graphs with vertex set V and suppose that R is also C_5 -free. If $G(B, R)$ is the complete graph, then V is obedient.*

Before we continue. We need the following definition.

Definition 1.6. We say that $G(B, R)$ contains a **double** C_5 if there is a set X of five vertices such that $B|X$ and $R|X$ are both induced C_5 in B and R , where $B|X$ (or $R|X$) denotes the subgraph of B (or R) induced by X .

It is clear that if B or R is C_5 -free, then $G(B, R)$ does not contains a double C_5 . Very recently, Gyárfás and Lehel generalized Theorem 1.5 as follow.

Theorem 1.7 (Theorem 3 of [4]). *Let B and R be C_4 -free graphs with vertex set V , and suppose that $G(B, R)$ does not contains a double C_5 . If $G(B, R)$ is the complete graph, then V is obedient.*

In section 2, we start with some basic definitions and prove the following theorem which generalize Lemma 3.4 of [1].

Theorem 1.8. *Let B and R be two graphs with vertex set V . Assume that the following hold:*

- B and R are C_4 -free graphs.
- $G(B, R)$ does not contains a double C_5 .
- $G(B, R)$ is the complete graph.
- Every proper subset $U \subset V$ is obedient.

If $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ is an induced C_5 in R , where $v_1 v_2, v_3 v_4 \in E(R \setminus B)$, then V is obedient.

In section 3, we use Theorem 1.8 to give another proof of Theorem 1.7, which we found independently. In section 4, we generalize Theorem 1.7 by proving the following.

Theorem 1.9. *Let B and R be two C_4 -free graphs on the same vertex set V and assume that $G(B, R)$ is the complete graph. Then there exists an B -clique X , an R -clique Y and a clique Z in B and R , such that $V = X \cup Y \cup Z$. Further, if $x \in Z$ then x is one of the vertices of some double C_5 in $G(B, R)$.*

As a corollary of Theorem 1.9, we obtain the following theorems.

Theorem 1.10. *If B and R are two C_4 -free graphs on the same vertex set V , then*

$$\omega(G(B, R)) \leq \omega(B) + \omega(R) + \omega(H(B, R)).$$

Theorem 1.11. *If B and R are two C_4 -free graphs on the same vertex set V then*

$$\omega(G(B, R)) \leq \omega(B) + \omega(R) + \frac{1}{2} \min(\omega(B), \omega(R)).$$

2. OBEDIENT SETS

The main result of this section is Theorem 2.6 which proved in [1] when B is C_5 -free (see Lemma 3.4 of [1]). Using a similar argument, we generalize it when $G(B, R)$ does not contains a double C_5 . First, we recall some basic definitions. Let $G = (V, E)$ be a graph.

Definition 2.1. An **induced subgraph** on a subset S of V , denoted by $G|S$, is a graph whose vertex set is S and whose edge set is $\{uv \mid u, v \in S \text{ and } uv \in E\}$.

Definition 2.2. For two disjoint subsets X and Y of V , we say that X is **G -complete (anticomplete)** to Y if every vertex of X is adjacent (non-adjacent) to every vertex of Y .

Let B and R be two graphs with vertex set V . We denote by $B \setminus R$ the graph with vertex set V such that two vertices are adjacent in $B \setminus R$ if and only if they are adjacent in B and non-adjacent in R .

Definition 2.3. • A set $C \subset V$ is a **cutset** if there exist disjoint nonempty $P, Q \subset V$ such that $V \setminus C = P \cup Q$ and P is anticomplete to Q in G .
 • A set $C \subset V$ is a **clique cutset** if it is a cutset and C is a clique of G .
 • A **weak clique cutset** in B is a clique C of B that is a cutset in $B \setminus R$.

Definition 2.4. For a graph H we say that G **contains** H if some induced subgraph of G is isomorphic to H . We say that G is **H -free** if G is not contains H .

Before we prove Theorem 2.6, we need the following useful theorem.

Theorem 2.5. (Lemma 3.2 of [1])

Let B and R be C_4 -free graphs with vertex set V , and assume that every proper subset $U \subset V$ is obedient. Assume also that $G(B, R)$ is a complete graph. If there is a weak clique cutset in B (or in R), then V is obedient.

Theorem 2.6. *Let B and R be two graphs with vertex set V . Assume that the following hold:*

- B and R are C_4 -free graphs.
- $G(B, R)$ does not contains a double C_5 .
- $G(B, R)$ is the complete graph.
- Every proper subset $U \subset V$ is obedient.

If $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ is an induced C_5 in R , where $v_1v_2, v_3v_4 \in E(R \setminus B)$, then V is obedient.

Proof. Since $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ is an induced C_5 , it follows that every edge in the cycle $v_1 - v_3 - v_5 - v_2 - v_4 - v_1$ belongs to $E(B \setminus R)$. If v_2v_3 is a blue edge, then $v_1 - v_4 - v_2 - v_3 - v_1$ is an induced C_4 in B and this contradict the assumption of the theorem. We obtain that $v_2v_3 \in E(R \setminus B)$. Since $G(B, R)$ does not contains a double C_5 , it follows that at least one of the edges v_1v_5, v_5v_4 belongs to $E(B)$. We may assume that $v_1v_5 \in E(B)$. Consider the cycle $v_1 - v_4 - v_2 - v_5 - v_1$. If v_4v_5 is not blue, then this cycle is an induced C_4 in B , a contradiction. It follows that v_4v_5 is a blue edge. We conclude that $\{v_1, v_2, v_3, v_4, v_5\}$ is an obedient set, where $\{v_1, v_4, v_5\}$ is the blue clique and $\{v_2, v_3\}$ is the red clique. If $V = \{v_1, v_2, v_3, v_4, v_5\}$, then we are done. Let $u \in V \setminus \{v_1, v_2, v_3, v_4, v_5\}$.

Claim: u is R -complete to $\{v_2, v_3\}$ or B -complete to $\{v_1, v_4, v_5\}$.

Proof of the claim: If u is R complete to $\{v_2, v_3\}$, then we are done. Assume that u is not R -complete to $\{v_2, v_3\}$ and without loss of generality, assume that $uv_2 \in E(B \setminus R)$. We show that in this condition u is B -complete to $\{v_1, v_4, v_5\}$. Assume in contrary that $uv_1 \in E(R \setminus B)$. Since R is C_4 -free, it follows that $uv_3 \in E(B \setminus R)$. For otherwise we have that $u - v_3 - v_2 - v_1 - u$ is an induced C_4 in R . Since B is C_4 -free, it follows that $uv_4 \in E(R \setminus B)$. For otherwise we have that $u - v_4 - v_1 - v_3 - u$ is an induced C_4 in B . It follows that $v_4 - v_2 - u - v_3 - v_1$ is a double C_5 in $G(B, R)$, a contradiction. So uv_1 is a blue edge.

Since B is C_4 -free, it follows that $uv_4 \in E(B)$. For otherwise we obtain that $uv_4 \in E(R \setminus B)$ and we have that $u - v_2 - v_4 - v_1 - u$ is an induced C_4 in B .

Since B is C_4 -free, it follows that $uv_5 \in E(B)$. For otherwise we have that $u - v_2 - v_5 - v_1 - u$ is an induced C_4 in B . We conclude that u is B -complete to $\{v_1, v_4, v_5\}$. Thus, we proved the claim.

Let A be the set of vertices in $V \setminus \{v_1, \dots, v_5\}$ that are not R -complete to $\{v_2, v_3\}$. Then A is B -complete to $\{v_1, v_4, v_5\}$.

Claim: A is an B -clique.

Proof of the claim: Let $u, v \in A$ and assume that $uv \in E(R \setminus B)$. Since u is not R -complete to $\{v_2, v_3\}$, we may assume that $uv_2 \in E(B \setminus R)$. We show that $vv_2 \in E(R)$. If $vv_2 \notin E(R)$, then $v - v_2 - u - v_1 - v$ is an induced C_4 in B , a contradiction. We conclude that $vv_2 \in E(R)$ and so $vv_3 \notin E(R)$ because $v \in A$. Similarly we obtain that $uv_3 \in E(R)$. But now $u - v - v_2 - v_3 - u$ is a C_4 in R , a contradiction. This proves the claim that A is a B -clique.

By the definition of A and the pervious claim we conclude that that $A \cup \{v_1, v_4, v_5\}$ is a B -clique. Let $Z = V \setminus (A \cup \{v_1, v_4, v_5\})$. If $Z = \{v_2, v_3\}$ then V is obedient. Assume that $Z \setminus \{v_2, v_3\} \neq \emptyset$. Note that every vertex in $Z \setminus \{v_2, v_3\}$ is R -complete to $\{v_2, v_3\}$ and so $Z \setminus \{v_2, v_3\}$ is anticomplete to $\{v_2, v_3\}$ in the graph $B \setminus R$. It follows that $A \cup \{v_1, v_4, v_5\}$ is a weak clique cutset in B . By Theorem 2.5, V is obedient. \square

3. STRUCTURES WITHOUT DOUBLE C_5

In this section, we give another proof of the recent result by Gyárfás and Lehel, which we found independently.

Theorem 3.1. *Let B and R be C_4 -free graphs with vertex set V , and assume that $G(B, R)$ is the complete graph. If $G(B, R)$ does not contains a double C_5 , then V is obedient.*

Proof. We use a similar argument to that in [3] (see the proof of Theorem 3). We prove the Theorem by induction on $|V|$. The theorem hold for $|V| = 1$ or $|V| = 2$. Assume that it is true for $|V| = 1, 2, \dots, n$ and let $|V| = n + 1$. Let $p \in V$. By the induction hypothesis $V \setminus \{p\}$ is obedient, namely there exist $X, Y \subseteq V$ such that $V \setminus \{p\} = X \cup Y$, X is a red clique and Y is a blue clique. We define the following sets

$$X_b = \{a \in X \mid ap \in E(B \setminus R)\} \text{ and } Y_r = \{a \in Y \mid ap \in E(R \setminus B)\}.$$

We may assume that X and Y is chosen with $|X_b| + |Y_r|$ minimum. If $|X_b| = 0$ or $|Y_r| = 0$, then the Theorem holds. Assume that $|X_b| \neq 0$ and $|Y_r| \neq 0$.

Let $q \in Y_r$. We claim that in the graph $B \setminus R$, q is connected to a vertex in X_b . If q is connected in red to all vertices of X , then $X \cup \{q\}$ is a red clique and $V \setminus \{p\} = (X \cup \{q\}) \cup Y \setminus \{q\}$ with $|(X \cup \{q\})_b| + |(Y \setminus \{q\})_r| < |X_b| + |Y_r|$, a contradiction to the minimality of $|X_b| + |Y_r|$. It follows that there is $r \in X$ such that $qr \in E(B \setminus R)$. If $r \in X_b$, then we are done. Suppose that $r \notin X_b$. So pr is a red edge. Let $s \in X_b$. If qs is a red edge then $p - q - s - r - p$ is an C_4 induced cycle in R , a contradiction. So $qs \in E(B \setminus R)$ and the claim follows. Similarly, for every $u \in X_b$ there exists $v \in Y_r$ such that $uv \in E(R \setminus B)$.

Let $x_1 \in Y_r$. There is $x_2 \in X_b$ such that $x_1x_2 \in E(B \setminus R)$. Also there is $x_3 \in Y_r$ such that $x_2x_3 \in E(R \setminus B)$. We continue in this way and obtain an even cycle $x_1 - x_2 - x_3 - x_4 - \dots$ such that $x_{2j-1} \in Y_r$, $x_{2j} \in X_b$, $x_{2j-1}x_{2j} \in E(B \setminus R)$ and $x_{2j}x_{2j+1} \in E(R \setminus B)$ for all $j \geq 1$. Let $x_1 - x_2 - x_3 - x_4 - \dots - x_k - x_1$ be the shortest even cycle that we can get in this way.

Case 1: $k=4$.

Since B is C_4 -free, it follows that $x_2x_4 \in E(R \setminus B)$. For otherwise we have that $x_1 - x_2 - x_4 - x_3 - x_1$ is an induced C_4 in B . Since R is C_4 -free, it follows that $x_1x_3 \in E(B \setminus R)$. For otherwise we have that $x_1 - x_4 - x_2 - x_3 - x_1$ is an induced C_4 in R . It follows that $p - x_3 - x_2 - x_4 - x_1 - p$ is a double C_5 in $G(B, R)$, a contradiction.

Case 2: $k > 4$.

By the minimality of k we obtain that x_1x_4 is an edge in B and R . Since R is C_4 -free, it follows that $x_1x_3 \in E(B \setminus R)$. For otherwise we have that $x_1 - x_4 - x_2 - x_3 - x_1$ is an induced C_4 in R . It follows that $x_2 - x_3 - p - x_1 - x_4 - x_2$ is an induced C_5 in R with $x_2x_3, px_1 \in E(R \setminus B)$. By Theorem 2.6, V is obedient. \square

Remark 3.2. Another way to prove Theorem 3.1 is to follow the proof of Theorem 1.7 in [1] and use Theorem 2.6 in every place the authors used Lemma 3.4 of [1].

4. STRUCTURES WITH DOUBLE C_5

In this section, we prove that if B and R are two C_4 -free graphs on the same vertex set V and $G(B, R)$ is the complete graph, then there exists an B -clique X , an R -clique Y and a clique Z in B and R , such that $V = X \cup Y \cup Z$. Further, if $x \in Z$ then x is one of the vertices of some double C_5 in $G(B, R)$. This generalize the recent result by Gyárfás and Lehel (Theorem 3.1). We begin with the following lemmas.

Lemma 4.1. *Let B and R be two C_4 -free graphs and $C : v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ be a double C_5 in $G(B, R)$ with blue edges and red diagonals. Assume that $G(B, R)$ is the complete graph and $x \in V \setminus \{v_1, \dots, v_5\}$. Then one of the following holds:*

- (1) x is connected in B to all the vertices of C .
- (2) x is connected in R to all the vertices of C .
- (3) there exists i such that
 - xv_i is an edge in B and R .
 - In $B \setminus R$: x is connected to the two neighbors of v_i in C .

- In $R \setminus B$: x is connected to the two neighbors of v_i in C .
In this case we say that xv_i is the **shared edge** of x .

Proof. Let $T = \{v_1, \dots, v_5, x\}$. If $\deg_{B|T}(x) = 0$, then condition (2) holds. Assume that $\deg_{B|T}(x) = 1$. Without loss of generality, assume that $xv_1 \in E(B)$. Since $G(B, R)$ is the complete graph, it follows that $xv_2, \dots, xv_5 \in E(R)$. Since R is C_4 -free, it follows that $xv_1 \in E(R)$. For otherwise we have that $x - v_3 - v_1 - v_4 - x$ is an induced C_4 in R . So $\deg_{R|T}(x) = 5$ and condition (2) holds. Assume that $\deg_{B|T}(x) = 2$. If $xv_i, xv_j \in E(B)$, where $v_i v_j \in E(R \setminus B)$, then $x - v_i - v_m - v_j - x$ is an induced C_4 in B , where v_m is the common neighbor of v_i and v_j in the cycle $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$, a contradiction. It follows that x is connected to two successive vertices of the cycle $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$. Without loss of generality, assume that $xv_1, xv_2 \in E(B)$. It follows that $xv_3, xv_4, xv_5 \in E(R)$. Since R is C_4 -free, it follows that $xv_1 \in E(R)$. For otherwise we have that $x - v_3 - v_1 - v_4 - x$ is an induced C_4 in R . Since R is C_4 -free, it follows that $xv_2 \in E(R)$. For otherwise we have that $x - v_4 - v_2 - v_5 - x$ is an induced C_4 in R . So $\deg_{R|T}(x) = 5$ and condition (2) holds. Assume that $\deg_{B|T}(x) = 3$. A similar argument shows that x is connected to three successive vertices of the cycle. Without loss of generality, assume that $xv_1, xv_2, xv_3 \in E(B)$. It follows that $xv_4, xv_5 \in E(R)$. Since R is C_4 -free, it follows that $xv_2 \in E(R)$. For otherwise we have that $x - v_4 - v_2 - v_5 - x$ is an induced C_4 in R . If $xv_1 \notin E(R)$ and $xv_3 \notin E(R)$, then condition (3) holds. If $xv_1 \in E(R)$, then $xv_3 \in E(R)$. For otherwise, we have that $x - v_1 - v_3 - v_5 - x$ is an induced C_4 in R . So condition (2) holds. Similarly, if $xv_3 \in E(R)$, then condition (2) holds. Assume that $\deg_{B|T}(x) = 4$. Without loss of generality, assume that $xv_1, xv_2, xv_3, xv_4 \in E(B)$. Since B is C_4 -free, it follows that $xv_5 \in E(B)$. For otherwise we have that $x - v_1 - v_5 - v_4 - x$ is an induced C_4 in B , a contradiction. So $\deg_{B|T}(x) = 5$ and condition (1) holds. \square

Lemma 4.2. *Let H be a C_4 -free graph and $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ be an induced C_5 cycle in H . If $x \neq y \notin \{v_1, \dots, v_5\}$ are two vertices connected to the same three successive vertices of the cycle, then $xy \in E(H)$.*

Proof. Without loss of generality, assume that $\{x, y\}$ is H -complete to $\{v_1, v_2, v_3\}$. If $xy \notin E(H)$, then $v_1 - y - v_3 - x - v_1$ is an induced C_4 in H , a contradiction. It follows that $xy \in E(H)$. \square

Now, we prove the main result.

Theorem 4.3. *Let B and R be two C_4 -free graphs on the same vertex set V and assume that $G(B, R)$ is the complete graph. Then there exists an B -clique X , an R -clique Y and a clique Z in B and R , such that $V = X \cup Y \cup Z$. Further, if $x \in Z$ then x is one of the vertices of some double C_5 in $G(B, R)$.*

Proof. If $G(B, R)$ does not contain a double C_5 , then by Theorem 3.1 there exists an B -clique X , an R -clique Y such that $V = X \cup Y$. By choosing $Z = \emptyset$, we are done. So assume that $G(B, R)$ contains a double C_5 : $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ with blue edges and red diagonals. If $V = \{v_1, \dots, v_5\}$, then by taking $X = \{v_1, v_2\}$, $Y = \{v_3, v_5\}$ and $Z = \{v_4\}$, we finish the proof. So assume that $V \setminus \{v_1, \dots, v_5\} \neq \emptyset$. We define the following sets:

$$M = \{p \mid p \notin \{v_1, \dots, v_5\} \text{ and } pv_i \in E(B) \text{ for all } 1 \leq i \leq 5\},$$

$$N = \{p \mid p \notin \{v_1, \dots, v_5\} \text{ and } pv_i \in E(R) \text{ for all } 1 \leq i \leq 5\},$$

$$A_j = \{p \mid p \notin \{v_1, \dots, v_5\} \text{ and } pv_j \text{ is the shared edge of } p\} \text{ for all } 1 \leq j \leq 5.$$

By Lemma 4.1, we have $V \setminus \{v_1, \dots, v_5\} = M \cup N \cup A_1 \cdots \cup A_5$.

If $p_1, p_2 \in M$, then $p_1v_i \in E(B)$ and $p_2v_i \in E(B)$ for all $1 \leq i \leq 5$. In particular, p_1 and p_2 are connected in B to the same three successive vertices of the induced cycle $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$. Since B is C_4 -free, by Lemma 4.2 we conclude that $p_1p_2 \in E(B)$. It follows that M is an B -clique. Similarly, if $p_1, p_2 \in N$, then p_1 and p_2 are connected in R to the same three successive vertices of the induced cycle $v_1 - v_3 - v_5 - v_2 - v_4 - v_1$. By Lemma 4.2, we obtain that N is an R -clique.

Let $p_1, p_2 \in A_1$. Note that $\{p_1, p_2\}$ is B -complete to $\{v_1, v_2, v_5\}$ and v_1, v_2, v_5 are successive vertices in the induced cycle $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ in B . By Lemma 4.2, we obtain that $p_1p_2 \in E(B)$. Note also that $\{p_1, p_2\}$ is R -complete to $\{v_1, v_3, v_4\}$ and v_1, v_3, v_4 are successive vertices in the induced cycle $v_1 - v_3 - v_5 - v_2 - v_4 - v_1$ in R . By Lemma 4.2, we obtain that $p_1p_2 \in E(R)$. It follows that A_1 is a clique in B and R . Similarly, A_j is a clique in B and R for all $2 \leq j \leq 5$.

A similar argument shows that $M \cup A_j$ is an B -clique for all $1 \leq j \leq 5$ and $N \cup A_j$ is an R -clique for all $1 \leq j \leq 5$.

Claim: $M \cup A_1 \cup A_2 \cup \{v_1, v_2\}$ is a clique in B .

Proof of the claim: If $x \in M \cup A_1$, then by the definition of M and A_1 , we obtain that $xv_1 \in E(B)$. If $x \in A_2$, then xv_2 is the shared edge of x . Since v_1 is a neighbor (in $B \setminus R$) of v_2 in the induced cycle $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$, it follows that $xv_1 \in E(B)$. So $\{v_1\}$ is B -complete to $M \cup A_1 \cup A_2$. Similarly, $\{v_2\}$ is B -complete to $M \cup A_1 \cup A_2$. We finish the proof of the claim if we show that A_1 is B -complete to A_2 . Let $x_1 \in A_1$ and $x_2 \in A_2$. If $x_1x_2 \in E(R)$, then $x_1 - x_2 - v_5 - v_3 - x_1$ is an induced C_4 in R , a contradiction. Since $G(B, R)$ is the complete graph it follows that $x_1x_2 \in E(B)$. Thus, we proved the claim.

Claim: $N \cup A_3 \cup A_5 \cup \{v_3, v_5\}$ is a clique in R .

Proof of the claim: If $x \in N \cup A_3$, then by the definition of N and A_3 , we obtain that $xv_3 \in E(R)$. If $x \in A_5$, then xv_5 is the shared edge of x . Since v_3 is a neighbor (in $R \setminus B$) of v_5 in the induced cycle $v_1 - v_3 - v_5 - v_2 - v_4 - v_1$, it follows that $xv_3 \in E(R)$. So $\{v_3\}$ is R -complete to $N \cup A_3 \cup A_5$. Similarly, $\{v_5\}$ is R -complete to $N \cup A_3 \cup A_5$. We finish the proof of the claim if we show that A_3 is R -complete to A_5 . Let $x_3 \in A_3$ and $x_5 \in A_5$. If $x_3x_5 \in E(B)$, then $x_3 - v_2 - v_1 - x_5 - x_3$ is an induced C_4 in B , a contradiction. Since $G(B, R)$ is the complete graph it follows that $x_3x_5 \in E(R)$. Thus, we proved the claim.

Claim: $A_4 \cup \{v_4\}$ is a clique in B and in R .

Proof of the claim: If $x \in A_4$, then xv_4 is the shared edge to x . So xv_4 is an edge in B and R . Thus, the claim follows from the definition of A_4 .

We set

$$X = M \cup A_1 \cup A_2 \cup \{v_1, v_2\}, \quad Y = N \cup A_3 \cup A_5 \cup \{v_3, v_5\}, \quad Z = A_4 \cup \{v_4\}.$$

By the above claims and Lemma 4.1, we obtain that X is a clique in B , Y is a clique in R and Z is a clique in B and R , with $V = X \cup Y \cup Z$.

Let $x \in Z$. If $x = v_4$, then $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ is a double C_5 that contains v_4 as a vertex. If $x \neq v_4$, then $x - v_5 - v_1 - v_2 - v_3 - x$ is a double C_5 in $G(B, R)$. Hence, if $x \in Z$ then x is one of the vertices of some double C_5 in $G(B, R)$. \square

As a corollary of Theorem 4.3, we obtain the following.

Corollary 4.4. *If B and R are two C_4 -free graphs on the same vertex set V then*

$$\omega(G(B, R)) \leq \omega(B) + \omega(R) + \omega(H(B, R)).$$

Proof. Let T be a maximum clique in $G(B, R)$. So $B|T$ and $R|T$ are two C_4 -free graphs on the same vertex set T such that $G(B|T, R|T)$ is the complete graph. By Theorem 4.3, there exists an $B|T$ -clique X , an $R|T$ -clique Y and a clique Z in $B|T$ and $R|T$, such that $T = X \cup Y \cup Z$. So

$$\omega(G(B, R)) = |T| \leq |X| + |Y| + |Z| \leq \omega(B) + \omega(R) + \omega(H(B, R)).$$

\square

Also, we have the following additional corollary of Theorem 4.3.

Corollary 4.5. *If B and R are two C_4 -free graphs on the same vertex set V then*

$$\omega(G(B, R)) \leq \omega(B) + \omega(R) + \frac{1}{2} \min(\omega(B), \omega(R)).$$

Proof. Let T be a maximum clique in $G(B, R)$. So $B|T$ and $R|T$ are two C_4 -free graphs on the same vertex set T such that $G(B|T, R|T)$ is the complete graph. If $G(B|T, R|T)$ does not contain a double C_5 , then by Theorem 3.1, T is the union of a clique in $B|T$ and a clique in $R|T$. It follows that $\omega(G(B, R)) \leq \omega(B) + \omega(R)$ and the theorem holds. Assume that $G(B|T, R|T)$ contains a double C_5 . Following the proof of Theorem 4.3, we may assume that A_4 is minimal so that $|A_4| \leq |A_i|$ for all $1 \leq i \leq 5$. We obtain that $2|Z| = 2|A_4| + 2 \leq |A_1| + |A_2| + |\{v_1, v_2\}| = |A_1 \cup A_2 \cup \{v_1, v_2\}| \leq \omega(B|T) \leq \omega(B)$. So $|Z| \leq \frac{1}{2}\omega(B)$. Similarly, $|Z| \leq \frac{1}{2}\omega(R)$. It follows that

$$\omega(G(B, R)) \leq \omega(B) + \omega(R) + \frac{1}{2} \min(\omega(B), \omega(R)).$$

\square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, MOUNT CARMEL, HAIFA 31905, ISRAEL
E-mail address: `abeer.othman@gmail.com`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, MOUNT CARMEL, HAIFA 31905, ISRAEL
E-mail address: `berger@math.haifa.ac.il`